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Coarse-grained equations of sound-wave propagation in two-phase random media. Renormalisation of physical properties

Takashi Nagatani†

Department of Applied Physics, Tokyo Institute of Technology, Meguroku, Tokyo 152, Japan

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Abstract. A general theory is presented to derive the coarse-grained equations governing the macroscopic behaviour of sound-wave propagation in suspensions from the standpoint of statistical continuum mechanics. The expressions for effective physical properties are obtained.

The resulting general expression for the effective sound-wave velocity is applied to acoustic wave propagations in bubbly fluid. It is shown that our result for the sound-wave velocity agrees well with experiment. We find that the basic equation for wave propagation in continuous random media does not apply to wave propagations in discrete random media (or suspensions). Our expressions for the effective viscosity and thermal conductivity agree with those for the effective viscosity and thermal conductivity in a dilute suspension in the dilute limit and are applied to more concentrated suspensions.

1. Introduction

In a study of sound-wave propagation in two-phase media (or suspensions), it is necessary to develop equations governing their mean field (van Wijingaarden 1972).

A number of phenomenological attempts have so far been made to draw up equations governing the macroscopic behaviour of disperse systems. However, there are no consistent theories for such systems which are satisfactory from a statistical point of view. Wave propagation theory in continuous random media applies to problems such as the scattering of sound waves by turbulent gases, the scattering of radio waves by tropospheric turbulence and the twinkling of stellar images. Considerable progress has been made by the use of formal perturbation methods with Green functions (Frisch 1968, Karal and Keller 1964, Howe 1971), but their applications to wave propagation in discrete random media (for example suspensions) have not been made. However, such formal methods have enjoyed considerable success in determining the effective thermal conductivity in suspensions (Beran 1968, Miller 1969, Hori 1977).

Because of the difficulty of analysing equations governing sound-wave propagations in suspensions by Green functions, it is not useful to apply this method to wave

† Present address: Department of Mechanical Engineering, Fukuoka University, Nishiku, Fukuokashi 814, Japan.

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propagation. In this paper, a new renormalisation method, which is similar to the perturbation method with Green functions, is applied to the problem of sound-wave propagation in suspensions.

If we consider suspensions as continua with variable physical properties, the governing equations will be stochastic differential equations with coefficients of random variables, because each phase has different values for the physical properties (Beran 1968). The equations governing the macroscopic (or coarse-grained) behaviour of sound-wave propagation in two-phase mixtures can be obtained by means of ensemble averaging on stochastic differential equations.

Ensemble averaging on equations can be replaced with the coarse graining of equations from the standpoint of statistical mechanics. Coarse graining is the eliminations of modes with high wavenumbers, which are comparable with the reciprocal of the size of the suspended particles, by looking at suspensions on a much larger scale than the size of the suspended particles. We consider that a sound wave with a wavelength much longer than the size of the suspended particles propagates in suspensions.

Then, by means of coarse graining, we can reduce the governing equations to equations with low modes which describe the behaviour on a much larger scale than the size of the suspended particles. We call the resulting equations governing macroscopic behaviour (or coarse-grained behaviour) the coarse-grained equations of sound-wave propagation.

This paper presents a method which gives coarse-grained equations for sound-wave propagations in two-phase media from the standpoint of statistical continuum mechanics.

In § 2, we derive a basic equation governing one-dimensional sound-wave propagation in a random stack of slabs which consists of two phases. It is shown that the basic equation of wave propagation in discrete random media is different from the equation of wave propagation in continuous random media. The derivation of the equations governing three-dimensional sound-wave propagation in suspensions is given in § 3. These equations are derived by the introduction of dissipative effects (viscosity and thermal conduction). In § 4, the procedure for coarse graining their equations is presented.

Then coarse-grained equations are derived and effective physical properties are obtained. The sound-wave velocity of two-phase mixtures is discussed in § 5. It is shown that our result for the sound velocity in a bubbly fluid agrees with experiment. In § 6, our expressions for the effective viscosity and thermal conductivity are discussed.

2. Equations governing one-dimensional sound-wave propagation

As a fundamental step for the study of sound-wave propagation in suspensions, for simplicity we consider a random stack of slabs which consists of two phases.

Suppose now that a small disturbance propagates in a two-phase random stack of slabs. For the purpose of describing the propagation of acoustic waves, we need to treat the linearised equations of motion.

Then the linearised equations of mass conservation are given by

$$(\partial\rho'/\partial t)+\rho_{\mathrm{L}}(\partial u'/\partial x)=0 \qquad \text{in one phase} \tag{1}$$

$$(\partial\rho'/\partial t)+\rho_{\mathrm{g}}(\partial u'/\partial x)=0 \qquad \text{in the other phase.} \tag{2}$$

If mass flux is continuous across the interfaces of the slabs, then, combining equations (1) and (2), the mass conservation equation in the random stack is given by

$$(\partial\rho'/\partial t)+\partial(\rho_0(x)u')/\partial x=0 \tag{3}$$

where the subscript 0 indicates the equilibrium state and primed quantities belong to small disturbances.

The density of equilibrium state $\rho_0(x)$ is a random variable which has density ρ_L in one phase and ρ_g in the other. However, we note that the fluid in a slab cannot permeate the interfaces, and while the velocity and pressure must be continuous at the slab interfaces, the mass flux need not be. Therefore equation (3) is incorrect for sound-wave propagation in a two-phase random stack of slabs.

The correct mass conservation equation combining equations (1) and (2) is given by

$$(\partial\rho'/\partial t)+\rho_0(x)(\partial u'/\partial x)=0. \tag{4}$$

The linearised equation of momentum conservation is given by

$$\rho_0(x)(\partial u'/\partial t)+(\partial p'/\partial x)=0 \tag{5}$$

where

$$\rho'(x,t)=((\partial\rho/\partial p)_0(x))p'(x,t).$$

The compressibility $((\partial\rho/\partial p)_0(x))$ is also a random variable with $(\partial\rho/\partial p)_L$ in one phase and $(\partial\rho/\partial p)_g$ in the other. Equations (4) and (5) represent stochastic differential equations with coefficients of random variables.

For the pressure p' we obtain from equations (4) and (5) that

$$\left[\left(\frac{1}{\rho}\frac{\partial\rho}{\partial p}\right)_0(x)\right]\frac{\partial^2 p'}{\partial t^2}-\frac{\partial}{\partial x}\left\{\left[\left(\frac{1}{\rho}\right)_0(x)\right]\frac{\partial p'}{\partial x}\right\}=0. \tag{6}$$

This equation gives a basic equation governing one-dimensional sound-wave propagation in a two-phase random stack of slabs. For reasons of comparison with equation (6), the equation which was derived by Howe (1971) in the statistical theory of sound-wave propagation in random media is given here:

$$\frac{\partial^2 p'}{\partial t^2}-\left[\left(\frac{\partial p}{\partial\rho}\right)_0(x)\right]\frac{\partial^2 p'}{\partial x^2}=0. \tag{7}$$

Equation (7) is derived using equation (3) in place of (1). Therefore (7) cannot be applied to wave propagation in suspensions although it gives a basic equation for wave propagation in continuous random media. We find that it is not justified to apply the basic equation for wave propagation in continuous random media to wave propagations in discrete random media such as suspensions.

In equation (6) the density and the compressibility are random parameters, while in (7) only the sound-wave velocity is a random parameter.

In § 5 we shall compare the effective sound velocity derived from equation (6) with the effective sound velocity from (7). It will be shown that there is a difference in the effective sound-wave velocities.

3. Equations governing three-dimensional sound-wave propagation in dissipative two-phase mixtures

In non-dissipative mixtures, the normal velocity and the pressure must be continuous at the interfaces. These conditions may be satisfied easily in one-dimensional sound-wave propagation by a stack of slabs, but it is difficult to satisfy the condition in three-dimensional sound-wave propagation by two-phase mixtures. For the derivation of equations governing three-dimensional sound-wave propagation in suspensions, we need to consider the effect of dissipations. Hence for the introduction of dissipative effects, not only the normal velocity but also the tangential velocity at the interfaces must be continuous in three-dimensional sound-wave propagation. Then it is not difficult to derive a complete set of equations which satisfy this condition.

Suppose now that deformable particles such as liquid drops are dispersed in the liquid so that they are statistically homogeneous and isotropic. Consider that a small disturbance propagates in this suspension. When we consider mixtures as continua with variable physical properties, physical quantities and currents such as velocity, pressure stress and heat flux must be continuous at the interface of the two phases. Under these conditions we obtain linearised equations governing three-dimensional sound-wave propagation in dissipative two-phase mixtures:

$$(\partial\rho'/\partial t)+\rho_0(\boldsymbol{x})(\partial u_i'/\partial x_i)=0 \tag{8}$$

$$\rho_0(\boldsymbol{x})\frac{\partial u_i'}{\partial t}+\frac{\partial p'}{\partial x_i}=\frac{\partial}{\partial x_j}\left[\mu(\boldsymbol{x})\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}-\frac{2}{3}\delta_{ij}\frac{\partial u_m'}{\partial x_m}\right)\right] \tag{9}$$

$$(\rho c_p)_0(\boldsymbol{x})\frac{\partial T'}{\partial t}-\frac{\partial p'}{\partial t}=\frac{\partial}{\partial x_i}\left(\lambda(\boldsymbol{x})\frac{\partial T'}{\partial x_i}\right) \tag{10}$$

$$\rho'=\left[\left(\frac{\partial\rho}{\partial T}\right)_{p_0}(\boldsymbol{x})\right]T'+\left[\left(\frac{\partial\rho}{\partial p}\right)_{T_0}(\boldsymbol{x})\right]p'. \tag{11}$$

Here, the density in the equilibrium state $\rho_0(\boldsymbol{x})$, the specific heat $(\rho c_p)_0(\boldsymbol{x})$, $(\partial\rho/\partial T)_{p_0}(\boldsymbol{x})$, $(\partial\rho/\partial p)_{T_0}(\boldsymbol{x})$, the viscosity $\mu(\boldsymbol{x})$ and thermal conductivity $\lambda(\boldsymbol{x})$ are random variables. These equations form a set of equations which govern the thermofluid dynamical behaviour of two phases in sound-wave propagation in dissipative two-phase mixtures. It will be difficult to analyse these equations directly because they are a set of stochastic equations.

We mostly require information on the macroscopic behaviour of disperse systems. Hence we want to derive equations governing macroscopic behaviour (or coarse-grained behaviour) of sound-wave propagation from equations (8)–(11).

In the following we will apply a renormalisation method to derive the coarse-grained equations.

4. Coarse graining of the basic equations

Instead of averaging equations (8)–(11), we use a coarse-graining procedure, which corresponds to dealing with wave-numbers sufficiently small compared with the reciprocal of the size of the suspended particles. The equations governing the macroscopic behaviour of a sound wave with a long wavelength can be obtained by coarse graining equations (8)–(11).

First, the physical properties are transformed as follows:

$$\begin{aligned}
\alpha(\boldsymbol{x}) &\equiv \rho_0(\boldsymbol{x}) = \alpha_{\mathrm{L}} + (\alpha_{\mathrm{g}} - \alpha_{\mathrm{L}})\xi(\boldsymbol{x}) \\
\beta(\boldsymbol{x}) &\equiv [(\partial p/\partial \rho)_{T_0}\rho_0](\boldsymbol{x}) = \beta_{\mathrm{L}} + (\beta_{\mathrm{g}} - \beta_{\mathrm{L}})\xi(\boldsymbol{x}) \\
\gamma(\boldsymbol{x}) &\equiv (\rho c_p)_0(\boldsymbol{x}) = \gamma_{\mathrm{L}} + (\gamma_{\mathrm{g}} - \gamma_{\mathrm{L}})\xi(\boldsymbol{x}) \\
\delta(\boldsymbol{x}) &\equiv [(\partial p/\partial \rho)_{T_0}(\partial \rho/\partial T)_{p_0}](\boldsymbol{x}) = \delta_{\mathrm{L}} + (\delta_{\mathrm{g}} - \delta_{\mathrm{L}})\xi(\boldsymbol{x}) \\
\mu(\boldsymbol{x}) &= \mu_{\mathrm{L}} + (\mu_{\mathrm{g}} - \mu_{\mathrm{L}})\xi(\boldsymbol{x}) \\
\lambda(\boldsymbol{x}) &= \lambda_{\mathrm{L}} + (\lambda_{\mathrm{g}} - \lambda_{\mathrm{L}})\xi(\boldsymbol{x})
\end{aligned} \tag{12}$$

where $\xi(\boldsymbol{x})$ is a normalised random variable which is equal to zero in a fluid phase and unity in dispersed phases.

Let us define the Fourier components $U_{i,\boldsymbol{k}}(\omega)$ of the velocity $u_i(\boldsymbol{x}, t)$ as

$$U_{i,\boldsymbol{k}}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}t \exp(\mathrm{i}\omega t) V^{-1/2} \int_{\mathrm{V}} \mathrm{d}\boldsymbol{x} \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}) u_i(\boldsymbol{x}, t) \tag{13}$$

$$u_i(\boldsymbol{x}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}t \exp(-\mathrm{i}\omega t) V^{-1/2} \sum_{\boldsymbol{k}'} \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}) U_{i,\boldsymbol{k}}(\omega). \tag{14}$$

We define the Fourier component of the pressure and the temperature in a similar way. Also, let us define the Fourier components $\xi_{\boldsymbol{k}}$ of the random variable $\xi(\boldsymbol{x})$ as

$$\xi_{\boldsymbol{k}} = V^{-1/2} \int_{\mathrm{V}} \mathrm{d}x \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}) \xi(\boldsymbol{x}) \tag{15}$$

$$\xi(\boldsymbol{x}) = V^{-1/2} \sum_{\boldsymbol{k}} \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}) \xi_{\boldsymbol{k}}. \tag{16}$$

Here V is the volume of mixtures. Substituting equations (14) and (16) into equations (8)–(11), we obtain the equations as follows:

$$\begin{aligned}
&\mathrm{i}\omega P_{\boldsymbol{k}} + \mathrm{i}\omega\delta_{\mathrm{L}} T_{\boldsymbol{k}} + \mathrm{i}k_i\beta_{\mathrm{L}} U_{i,\boldsymbol{k}} + V^{-1/2} \sum_{\boldsymbol{k}'} \mathrm{i}\omega(\delta_{\mathrm{g}} - \delta_{\mathrm{L}})\xi_{\boldsymbol{k}'} T_{(\boldsymbol{k}-\boldsymbol{k}')} \\
&\qquad + V^{-1/2} \sum_{\boldsymbol{k}'} \mathrm{i}(k_i - k'_i)(\beta_{\mathrm{g}} - \beta_{\mathrm{L}})\xi_{\boldsymbol{k}'} U_{i,(\boldsymbol{k}-\boldsymbol{k}')} = 0
\end{aligned} \tag{17}$$

$$\begin{aligned}
&\mathrm{i}\omega\alpha_{\mathrm{L}} U_{i,\boldsymbol{k}} + \mathrm{i}k_i P_{\boldsymbol{k}} + V^{-1/2} \sum_{\boldsymbol{k}'} \mathrm{i}\omega(\alpha_{\mathrm{g}} - \alpha_{\mathrm{L}})\xi_{\boldsymbol{k}'} U_{i,(\boldsymbol{k}-\boldsymbol{k}')} \\
&\qquad = -\mu_{\mathrm{L}}(k^2\delta_{ij} + \tfrac{1}{3}k_i k_j) U_{j,\boldsymbol{k}} - V^{-1/2} \sum_{\boldsymbol{k}'} [k_m(k_m - k'_m)\delta_{ij} + k_j(k_i - k'_i) \\
&\qquad\quad - \tfrac{2}{3}k_i(k_j - k'_j)](\mu_{\mathrm{g}} - \mu_{\mathrm{L}})\xi_{\boldsymbol{k}'} U_{j,(\boldsymbol{k}-\boldsymbol{k}')}
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\mathrm{i}\omega\gamma_{\mathrm{L}} T_{\boldsymbol{k}} - \mathrm{i}\omega P_{\boldsymbol{k}} + V^{-1/2} \sum_{\boldsymbol{k}'} \mathrm{i}\omega(\gamma_{\mathrm{g}} - \gamma_{\mathrm{L}})\xi_{\boldsymbol{k}'} T_{(\boldsymbol{k}-\boldsymbol{k}')} \\
&\qquad = -\lambda_{\mathrm{L}} k^2 T_{\boldsymbol{k}} - V^{-1/2} \sum_{\boldsymbol{k}'} k_i(k_i - k'_i)(\lambda_{\mathrm{g}} - \lambda_{\mathrm{L}})\xi_{\boldsymbol{k}'} T_{(\boldsymbol{k}-\boldsymbol{k}')}.
\end{aligned} \tag{19}$$

We note that the Fourier component $\xi_{\boldsymbol{k}}$ takes a non-zero value at the wavenumber $\boldsymbol{k} = 0$ and near the wavenumber k comparable with the reciprocal of the size of the suspended particles, because the random variable $\xi(\boldsymbol{x})$ varies on scale of the size of the suspended particles.

Since we consider the macroscopic behaviour of sound-wave propagation, we take the limit $|\boldsymbol{k}| \ll 2\pi/a$, where a represents the size of a suspended particle. Then the terms multiplied by $\xi_{k'}(\boldsymbol{k}' \neq 0)$ have high wavenumber $\boldsymbol{k}'$ compared with other terms, because the Fourier component $\xi_{k'}(\boldsymbol{k}' \neq 0)$ takes a non-zero value only near the wavenumber $\boldsymbol{k}'$ comparable with $2\pi/a$.

We can distinguish between terms with low wavenumbers and terms with high wavenumbers. The terms with low wavenumber $\boldsymbol{k}$ consist of physical quantities referrable to macroscopic motion, while the terms with high wavenumber $\boldsymbol{k}'$ consist of physical quantities referrable to microscopic motion on the scale of the size of the dispersed particle. It is necessary to eliminate modes with high wavenumber. By iteration we shall derive the coarse-grained equations, which consist of physical quantities referrable to macroscopic motions. Because we eliminate modes with high wavenumber iteratively, we derive the formal solutions of $U_{i,k}$, T_k from equations (17)–(19):

$$U_{i,k} = -V^{-1/2} \sum_{k'} \left[\frac{k_i(k_j - k'_j)}{k^2} \left(\frac{\beta_g - \beta_L}{\beta_L} \right) + \left(\frac{k_m(k_m - k'_m)}{k^2} \delta_{ij} + \frac{k_j(k_i - k'_i)}{k^2} - \frac{2k_i k_j k_m (k_m - k'_m)}{k^4} \right) \left(\frac{\mu_g - \mu_L}{\mu_L} \right) \right] \xi_{k'} U_{j,(k-k')} + O(\omega) \tag{20}$$

$$T_k = -V^{-1/2} \sum_{k'} \frac{k_i(k_i - k'_i)}{k^2} \left(\frac{\beta_g - \beta_L}{\beta_L} \right) \xi_{k'} T_{(k-k')} + O(\omega) \tag{21}$$

where we eliminated terms of $O(\omega)$ in the limit $\omega \to 0$, since the macroscopic behaviour of a sound wave with a long wavelength is slow compared with the fluctuation behaviour on a scale of the size of the suspended particles.

Then, successive substitutions of equations (20) and (21) in terms multiplied by $\xi_{k'}$ lead to

$$V^{-1/2} \sum_{k'} \mathrm{i}\omega \xi_{k'} T_{(k-k')} = \mathrm{i}\omega \langle \xi \rangle T_k - V^{-1} \sum_{\substack{k' \\ (k' \neq 0)}} \mathrm{i}\omega \frac{(k_i - k'_i)k_i}{(k-k')^2} \left(\frac{\lambda_g - \lambda_L}{\lambda_L} \right) (\xi_{k'} \xi_{-k'}) T_k + V^{-3/2} \sum_{\substack{k' \\ (k' \neq 0)}} \sum_{\substack{k'' \\ (k'' \neq -k')}} \mathrm{i}\omega \frac{(k_i - k'_i)(k_i - k'_i - k''_i)(k_j - k'_j - k''_j)k_j}{(k-k')^2 (k-k'-k'')^2} \times \left(\frac{\lambda_g - \lambda_L}{\lambda_L} \right)^2 (\xi_{k'} \xi_{k''} \xi_{-k'-k''}) T_k + \ldots \tag{22}$$

$$V^{-1/2} \sum_{k'} \mathrm{i}(k_i - k'_i) \xi_{k'} U_{i,(k-k')} = \mathrm{i}k_i \langle \xi \rangle U_{i,k} - V^{-1} \sum_{\substack{k' \\ (k' \neq 0)}} \mathrm{i} \left[\frac{(k_i - k'_i)k_j}{(k-k')^2} \left(\frac{\beta_g - \beta_L}{\beta_L} \right) + \left(\frac{(k_j - k'_j)(k_m - k'_m)k_m}{(k-k')^2} + \frac{(k_i - k'_i)(k_j - k'_j)k_i}{(k-k')^2} - \frac{2(k_i - k'_i)^2 (k_j - k'_j)(k_m - k'_m)k_m}{(k-k')^4} \right) \left(\frac{\mu_g - \mu_L}{\mu_L} \right) \right] (\xi_{k'} \xi_{-k'}) U_{j,k} + \ldots . \tag{23}$$

Because the particles are dispersed statistically homogeneously and isotropically in the liquid and the wavenumber $\boldsymbol{k}$ is sufficiently small compared with the reciprocal of the size of the suspended particles, the odd-order products of $(k_i - k_i')$ will be approximately zero and the even-order products of $(k_i - k_i')$ lead to the relationships

$$\begin{aligned}&(k_i - k_i')(k_j - k_j')/(k-k')^2 \simeq \tfrac{1}{3}\delta_{ij}\\&(k_i - k_i')k_j - k_j')(k_p - k_p')(k_q - k_q')/(k-k')^4 \simeq \tfrac{1}{15}(\delta_{ij}\delta_{pq} + \delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}).\end{aligned} \tag{24}$$

Then, making use of these relations and the inverse transform of the $\xi_{\boldsymbol{k}}$, equations (20) and (21) result in simple forms with only low wavenumbers $\boldsymbol{k}$ as follows:

$$V^{-1/2}\sum_{\boldsymbol{k}'} \mathrm{i}\omega\xi_{\boldsymbol{k}'}T_{(\boldsymbol{k}-\boldsymbol{k}')} = \mathrm{i}\omega\langle\xi\rangle T_{\boldsymbol{k}} \tag{25}$$

$$\begin{aligned}&V^{-1/2}\sum_{\boldsymbol{k}'} \mathrm{i}(k_i - k_i')\xi_{\boldsymbol{k}'}U_{i,(\boldsymbol{k}-\boldsymbol{k}')}\\&= \mathrm{i}k_i\langle\xi\rangle U_{i,\boldsymbol{k}} - \mathrm{i}k_i\frac{(\beta_{\mathrm{g}}-\beta_{\mathrm{L}})}{\beta_{\mathrm{L}}}(\langle\xi\rangle - \langle\xi\rangle^2)U_{i,\boldsymbol{k}}\\&\quad + \mathrm{i}k_i\frac{(\beta_{\mathrm{g}}-\beta_{\mathrm{L}})^2}{\beta_{\mathrm{L}}^2}(\langle\xi\rangle - 2\langle\xi\rangle^2 + \langle\xi\rangle^3)U_{i,\boldsymbol{k}} + \dots .\end{aligned} \tag{26}$$

Here we used the relationships

$$\begin{aligned}V^{-1}\sum_{\substack{\boldsymbol{k}'\\(\boldsymbol{k}'\neq 0)}} \xi_{\boldsymbol{k}'}\xi_{-\boldsymbol{k}'} &= \frac{1}{V}\int_{\mathrm{v}} \mathrm{d}\boldsymbol{x}\,\xi(\boldsymbol{x})^2 - \left(\frac{1}{V}\int_{\mathrm{v}}\mathrm{d}\boldsymbol{x}\,\xi(\boldsymbol{x})\right)^2\\&\equiv \langle\xi^2\rangle - \langle\xi\rangle^2\end{aligned} \tag{27}$$

$$\langle\xi^n\rangle = \langle\xi\rangle \tag{28}$$

where $\langle\ \rangle$ indicates the volume average, n being a positive integer. We find the coarse-grained version of equation (17):

$$\mathrm{i}\omega P_{\boldsymbol{k}} + \mathrm{i}\omega\delta^* T_{\boldsymbol{k}} + \mathrm{i}k_i\beta^* U_{i,\boldsymbol{k}} = 0 \tag{29}$$

$$\delta^* = \langle\delta\rangle \tag{30}$$

$$\begin{aligned}\beta^* &= \beta_{\mathrm{L}}\left[1 + \left(\frac{\beta_{\mathrm{g}}-\beta_{\mathrm{L}}}{\beta_{\mathrm{L}}}\right)\langle\xi\rangle - \left(\frac{\beta_{\mathrm{g}}-\beta_{\mathrm{L}}}{\beta_{\mathrm{L}}}\right)^2(\langle\xi\rangle - \langle\xi\rangle^2) + \left(\frac{\beta_{\mathrm{g}}-\beta_{\mathrm{L}}}{\beta_{\mathrm{L}}}\right)^3(\langle\xi\rangle - 2\langle\xi\rangle^2 + \langle\xi\rangle^3) + \dots\right]\\&= \beta_{\mathrm{L}}\left[1 + \left(\frac{\beta_{\mathrm{g}}-\beta_{\mathrm{L}}}{\beta_{\mathrm{g}}}\right)\langle\xi\rangle + \left(\frac{\beta_{\mathrm{g}}-\beta_{\mathrm{L}}}{\beta_{\mathrm{g}}}\right)^2\langle\xi\rangle^2 + \dots\right]\\&= 1/\langle 1/\beta\rangle\end{aligned} \tag{31}$$

where the asterisk indicates effective physical properties.

We also find coarse-grained versions of equations (18) and (19) in a similar way:

$$\mathrm{i}\omega\alpha^* U_{i,\boldsymbol{k}} + \mathrm{i}k_i P_{\boldsymbol{k}} = -\mu^*(k^2\delta_{ij} + \tfrac{1}{3}k_i k_j)U_{j,\boldsymbol{k}} \tag{32}$$

$$\mathrm{i}\omega\gamma^* T_{\boldsymbol{k}} - \mathrm{i}\omega P_{\boldsymbol{k}} = -\lambda^* k^2 T_{\boldsymbol{k}} \tag{33}$$

$$\alpha^* = \langle\alpha\rangle \tag{34}$$

$$\gamma^* = \langle\gamma\rangle \tag{35}$$

$$\mu^* = \mu_L\left[1+\left(\frac{\mu_g-\mu_L}{\mu_L}\right)\langle\xi\rangle-\left(\frac{\mu_g-\mu_L}{\mu_L}\right)^2(\tfrac{2}{5}\langle\xi\rangle-\langle\xi\rangle^2)\right.$$
$$\left.+\left(\frac{\mu_g-\mu_L}{\mu_L}\right)^3[(\tfrac{2}{5})^2\langle\xi\rangle-2(\tfrac{2}{5})\langle\xi\rangle^2+\langle\xi\rangle^3]-\ldots\right]$$
$$=\mu_L\left(\frac{1}{1+[5(\mu_L-\mu_g)/(3\mu_L+2\mu_g)]\phi}\right) \tag{36}$$

$$\lambda^* = \lambda_L\left[1+\left(\frac{\lambda_g-\lambda_L}{\lambda_L}\right)\langle\xi\rangle-\left(\frac{\lambda_g-\lambda_L}{\lambda_L}\right)^2(\tfrac{1}{3}\langle\xi\rangle-\langle\xi\rangle^2)+\ldots\right]$$
$$=\lambda_L\left(\frac{1}{1+[3(\lambda_L-\lambda_g)/(2\lambda_L+\lambda_g)]\phi}\right) \tag{37}$$

where we have used the relation

$$\langle\xi\rangle=\phi \tag{38}$$

and ϕ represents a volume concentration of dispersed phases.

Then, using the inverse transform, we obtain coarse-grained equations governing the macroscopic behaviour of sound-wave propagations in dissipative mixtures:

$$\frac{\partial p'}{\partial t}+\left\langle\left(\frac{\partial p}{\partial\rho}\right)_{T_0}\left(\frac{\partial\rho}{\partial T}\right)_{p_0}\right\rangle\frac{\partial T'}{\partial t}+\frac{1}{\langle(1/\rho_0)(\partial\rho/\partial p)_{T_0}\rangle}\frac{\partial u_i'}{\partial x_i}=0 \tag{39}$$

$$\langle\rho_0\rangle\frac{\partial u_i'}{\partial t}+\frac{\partial p'}{\partial x_i}=\mu^*\left(\nabla^2u_i'+\frac{1}{3}\frac{\partial^2u_j'}{\partial x_i\,\partial x_j}\right) \tag{40}$$

$$\langle(\rho c_p)_0\rangle\frac{\partial T'}{\partial t}-\frac{\partial p'}{\partial t}=\lambda^*\nabla^2T'. \tag{41}$$

We note that the coefficients of equations (39)–(41) represent effective values of physical properties. We find that coefficients of terms with time derivatives in their equations have the simple volume average, but coefficients of terms with space derivatives do not. By means of this renormalisation method, we can obtain all the effective values of physical properties. Similarly, we can also obtain the coarse-grained version of equation (6), governing a one-dimensional sound-wave propagation in a two-phase random stack of slabs. Such calculations are given in the appendix.

We note that the coarse-grained equations (39)–(41) do not have decaying plane-wave solutions for the mean field without dissipations, since we consider the macroscopic behaviour of a sound wave with a wavelength much longer than the size of a suspended particle. If we want to derive decaying plane-wave solutions for the average fields without dissipations, we need to calculate the wavevector and frequency-dependent effective physical properties. Such effective physical properties could perhaps be calculated if we do not neglect the terms of $O(\omega)$ in equations (20) and (21), but such calculations would be troublesome. For simplicity, we have treated the effective physical properties at zero wavevector and frequency. Next we shall derive the sound velocity from the coarse-grained equations (39)–(41) and compare our result with other work.

5. The velocity of sound in liquid–gas mixtures

We shall consider the sound velocity in liquid–gas mixtures which are representative of sound-wave propagation in two-phase mixtures.

The interesting phenomenon that the speed of sound in a bubbly fluid is lower than the speed of sound in pure gases is already known (van Wijingaarden 1972).

The characteristics of the sound velocity have been explained by non-statistical theories since statistical treatments of derivatives of the sound velocity have not been successful.

We will derive the sound velocity from the coarse-grained equations (39)–(41) governing the macroscopic behaviour of wave motions in suspensions. Because we obtain only the sound velocity, we neglect terms with the viscosity and the thermal conductivity in equations (40) and (41).

We obtain the equation for the pressure p' from the equations

$$(\partial^2 p'/\partial t^2) - c^{*2}(\partial^2 p'/\partial x_i^2) = 0 \tag{42}$$

$$c^{*2} = \left[\langle\rho_0\rangle\left\langle\frac{1}{\rho_0}\left(\frac{\partial\rho}{\partial p}\right)_{T_0}\right\rangle\left(1 + \left\langle\left(\frac{\partial p}{\partial\rho}\right)_{T_0}\left(\frac{\partial\rho}{\partial T}\right)_{p_0}\right\rangle\Big/\langle(\rho c_p)_0\rangle\right)\right]^{-1}. \tag{43}$$

The c^* in equation (43) gives an expression for the sound velocity in suspensions.

We apply equation (43) to a bubbly fluid. Neglecting ρ_g/ρ_L, $(\rho c_p)_g/(\rho c_p)_L$ and $((\partial p/\partial T)_{\rho_0})_L/((\partial p/\partial T)_{\rho_0})_g$ with respect to unity, we obtain the approximate sound velocity

$$c^{*2} \simeq p_0/[\rho_L\phi(1-\phi)]. \tag{44}$$

This result agrees with the sound velocity derived from the non-statistical treatment under the assumption of isothermal behaviour (van Wijingaarden 1972).

We described the acoustic wave propagation in a two-phase random stack of slabs in § 2, and we give the coarse graining procedure for equations (6) and (7) in the appendix. The sound velocities derived from equations (6) and (7) are given by

$$c_{(3)}^{*2} = \left(\langle\rho\rangle\left\langle\frac{1}{\rho_0}\left(\frac{\partial\rho}{\partial p}\right)_0\right\rangle\right)^{-1} \tag{45}$$

$$c_{(4)}^{*2} = \langle(\partial\rho/\partial p)_0\rangle^{-1}. \tag{46}$$

We apply these expressions for the sound velocity to a random stack of slabs which consists of the liquid and gas phases. Neglecting ρ_g/ρ_L with respect to unity and assuming isothermal behaviour for the gas, we obtain the approximate sound velocity from equation (45):

$$c_{(3)}^{*2} \simeq p_0/[\rho_L\phi(1-\phi)]. \tag{47}$$

Here, we need the assumption of isothermal behaviour for gases for one-dimensional sound-wave propagation in a non-dissipative mixture compounded of a two-phase random stack of slabs, but we do not need it for three-dimensional wave propagation in suspensions.

The sound velocity from equation (46) is given by

$$c_{(4)}^{*2} = \frac{1}{c_L^{-2}(1-\phi) + c_g^{-2}\phi} \tag{48}$$

where c_L and c_g are the sound velocities of liquid and gas respectively.

We note that the approximate sound velocities (44) and (47) cannot be applied in the limit $\phi = 0$ and $\phi = 1$. In this limit we must apply the expressions (43) and (45).

For a comparison between equations (6) and (7), we show the sound velocities (45) and (46) in figure 1. The sound velocity of equation (45) agrees with experiment (van Wijinhaarden 1972), but the sound velocity of equation (46) does not. Equation (7) cannot be applied to the sound-wave propagation in discrete random media.

The sound velocity c^* which is given by the relation (43) is a rigorous one derived from the standpoint of statistical continuum mechanics.

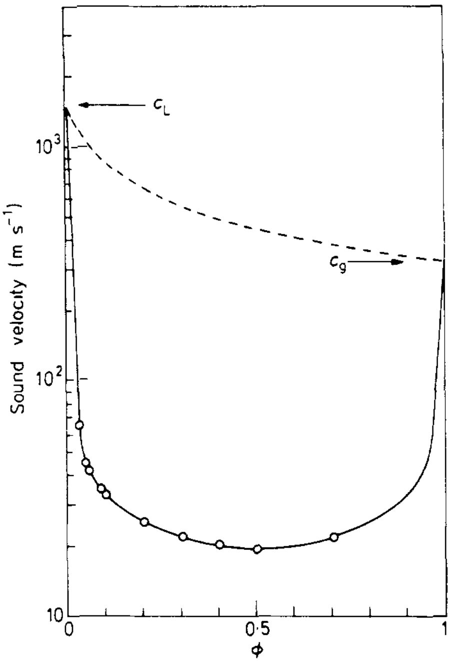

Figure 1. Plots of the effective sound-wave velocity c^* against the volume fraction ϕ. The full and broken curves represent equations (45) and (46) respectively and the open circles indicate the data of Karplus (1961).

6. Effective viscosity and effective thermal conductivity

The coarse-grained equations governing the macroscopic behaviour of sound-wave propagation have been given in equations (39)–(41).

This sound wave will be damped by the effect of dissipations such as effective viscosity and effective thermal conductivity. We think that effective transport coefficients in sound-wave propagation must agree with the effective transport coefficients of the other physical phenomena in suspensions, and therefore compare our effective transport coefficients with other results.

First, we discuss the effective viscosity. Einstein's formula for the effective viscosity in dilute suspensions of spherical particles is already known (Landau and Lifshitz 1960). Recently the coefficient of the ϕ^2 term has also been calculated for spheres by a variety of methods, often with rather different numerical results (Batchelor and Green 1972, Bedeaux *et al* 1977).

We expand the expression (36) for the effective viscosity in a series for the purpose of comparison:

$$\frac{\mu^*}{\mu_L} = 1 - \frac{5(\mu_L - \mu_p)}{(3\mu_L + 2\mu_p)}\phi + \frac{25(\mu_L - \mu_p)^2}{(3\mu_L + 2\mu_p)^2}\phi^2 - \frac{125(\mu_L - \mu_p)^3}{(3\mu_L + 2\mu_p)^3}\phi^3 + \dots \qquad (49)$$

where we use μ_L for the viscosity of the liquid phase and μ_p for the viscosity of suspended particles. The coefficient of the ϕ term agrees with that for a dilute suspension of deformable particles.

The expression found by Batchelor and Green for the effective viscosity in suspensions of rigid spherical particles is given by

$$\mu^*/\mu_L = 1 + \tfrac{5}{2}\phi + (7{\cdot}6 \mp 0{\cdot}8)\phi^2. \qquad (50)$$

That found by Bedeaux *et al* is given by

$$\mu^*/\mu_L = 1 + \tfrac{5}{2}\phi + 4{\cdot}8\phi^2. \qquad (51)$$

In the series (49), the limit $\mu_p \to \infty$ yields

$$\mu^*/\mu_L = 1 + \tfrac{5}{2}\phi + 6{\cdot}25\phi^2. \qquad (52)$$

Our expression (36) for deformable particles will be applied not only to a suspension with low concentrations but also to more concentrated suspensions.

Second, we discuss the effective thermal conductivity. The expression for the effective thermal conductivity to first order in ϕ was found by Maxwell and recently to second order in ϕ by Jeffrey (1973).

We expand the expression (37) for the effective thermal conductivity in a series:

$$\frac{\lambda^*}{\lambda_L} = 1 - \frac{3(\lambda_L - \lambda_p)}{(2\lambda_L + \lambda_p)}\phi + \frac{9(\lambda_L - \lambda_p)^2}{(2\lambda_L + \lambda_p)^2}\phi^2 - \frac{27(\lambda_L - \lambda_p)^3}{(2\lambda_L + \lambda_p)^3}\phi^3 + \dots \qquad (53)$$

where we use λ_L for the thermal conductivity of a matrix and λ_p for the thermal conductivity of suspended particles.

The coefficient of the ϕ term agrees with that for a dilute suspension of spherical particles. For the limit $\lambda_p \to \infty$, we find a value of 9 for the numerical coefficient of the ϕ^2 term. Our result is about twice the value found by Jeffrey.

We note that our results for the effective viscosity and the effective thermal conductivity cannot be applied to suspensions with high concentrations since there is a phase transition (cf percolation threshold) between $\phi = 0$ and $\phi = 1$ (Hori and Yonezawa 1977).

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Appendix.

In this appendix, we show the coarse-graining of equations (6) and (7) making use of the method in § 4.

First, we take the Fourier transform of equation (6) with respect to wavenumber k and frequency ω:

$$\alpha_L\omega^2 P_k - \beta_L k^2 P_k + (\alpha_g - \alpha_L)\omega^2 V^{-1/2}\sum_{k'} \xi_{k'} P_{(k-k')} - (\beta_g - \beta_L) V^{-1/2} \sum_{k'} k(k-k')\xi_{k'} P_{(k-k')} = 0 \tag{A.1}$$

where we have used the following definitions:

$$\alpha(x) \equiv \left(\frac{1}{\rho}\frac{\partial\rho}{\partial p}\right)_0 (x) = \alpha_L + (\alpha_g - \alpha_L)\xi(x)$$

$$\beta(x) \equiv (1/\rho)_0(x) = \beta_L + (\beta_g - \beta_L)\xi(x).$$

We derive the equation for P_k from equation (A.1):

$$P_k = -V^{-1/2}\sum_{k'}\left(\frac{(\alpha_g - \alpha_L)\omega^2 - (\beta_g - \beta_L)k(k-k')}{\alpha_L\omega^2 - \beta_L k^2}\right)\xi_{k'} P_{(k-k')}. \tag{A.2}$$

Then, we eliminate modes with high wavenumber by successive substitution of equation (A.2) in the terms multiplied by $\xi_{k'}$ and we obtain the following equation:

$$\alpha_L\left[1 + \frac{(\alpha_g - \alpha_L)}{\alpha_L}\langle\xi\rangle - V^{-1}\sum_{\substack{k' \\ (k'\neq 0)}}\left(\frac{(\alpha_g - \alpha_L)\omega^2 - (\beta_g - \beta_L)k(k-k')}{\alpha_L\omega^2 - \beta_L(k-k')^2}\right)\left(\frac{\alpha_g - \alpha_L}{\alpha_L}\right)\right.$$
$$\left.\times(\xi_{k'}\xi_{-k'}) + \ldots\right]\omega^2 P_k + \beta_L\left[k^2 + k^2\left(\frac{\beta_g - \beta_L}{\beta_L}\right)\langle\xi\rangle - V^{-1}\sum_{\substack{k' \\ (k'\neq 0)}} k(k-k')\right.$$
$$\left.\times\left(\frac{(\alpha_g - \alpha_L)\omega^2 - (\beta_g - \beta_L)k(k-k')}{\alpha_L\omega^2 - \beta_L(k-k')^2}\right)\left(\frac{\beta_g - \beta_L}{\beta_L}\right)(\xi_{k'}\xi_{-k'}) + \ldots\right]P_k = 0. \tag{A.3}$$

In the limit $|k| \ll |k'|$, equation (A.3) results in the simple form

$$\alpha_L\left[1 + \left(\frac{\alpha_g - \alpha_L}{\alpha_L}\right)\langle\xi\rangle\right]\omega^2 P_k - \beta_L\left[1 + \left(\frac{\beta_g - \beta_L}{\beta_L}\right)\langle\xi\rangle - \left(\frac{\beta_g - \beta_L}{\beta_L}\right)^2(\langle\xi\rangle - \langle\xi\rangle^2) + \ldots\right]k^2 P_k = 0. \tag{A.4}$$

Using the inverse transform, we obtain the coarse-grained equation governing a mean field of the random stack of slabs:

$$\left\langle\left(\frac{1}{\rho}\frac{\partial\rho}{\partial p}\right)_0\right\rangle\frac{\partial^2 p'}{\partial t^2} - \frac{1}{\langle\rho_0\rangle}\frac{\partial^2 p'}{\partial x^2} = 0. \tag{A.5}$$

We obtain the coarse-grained version of equation (7) in a similar way:

$$\frac{\partial^2 p'}{\partial t^2} - \frac{1}{\langle(\partial\rho/\partial p)_0\rangle}\frac{\partial^2 p'}{\partial x^2} = 0. \tag{A.6}$$

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